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STANDARDIZED TESTS AND THE IMPROVEMENT OF TEACHING IN FIRST-YEAR ALGEBRA[1]

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D. THE FUNCTION OF ALGEBRA TESTS

Algebra tests may have two functions: (1) they provide a yardstick for measuring the results of instruction and for determining the relative efficiency of teaching and the progress that teachers and pupils are making; (2) *their more important function, however, is that of diagnosis: they reveal difficulties in learning and point out needed changes in teaching emphasis.*

1. MEASURING THE RESULTS OF INSTRUCTION IN ALGEBRA IN 27 SCHOOL SYSTEMS BY MEANS OF THE TESTS

The formal Tests I–XI have been given in the following 27 school systems:

Cities in Illinois

*Rossville
Robinson
Bradley
Pekin
*Paris
*Palatine
*Oak Park
*Mount Olive
*Mount Carmel
Marshall
*Lincoln
*Highland Park
*Henry
Gibson City
*Divernon
*Chicago Heights
Charleston High School
Charleston Normal School
*Carbondale Normal School
Cairo
*Benton
*Watseka
Waukegan
*Decatur
*Joliet

Cities Outside Illinois

Muskogee, Okla.
*Mankato, Minn.

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[1] A final report on the "Experimental Determination of Standards in First Year Algebra," made to the Mathematics Section of the University of Illinois High-School Conference, November 24, 1916. Part I appeared in *School Review*, February, 1917.

In those 17 cities marked with an asterisk the verbal tests were also given. In addition to these 17 cities the verbal tests were also given in the following four schools:

Crane Technical High School, Chicago
Parker High School, Chicago
Englewood High School, Chicago
Carter Harrison High School, Chicago

and in four schools which returned papers with no names. The number of pupils taking the tests in each school was about 50. The total number of pupils solving the formal tests was 1,182; the total number solving the verbal tests was 1,295.

From the detailed records of the results of this testing in 27 school systems we give in Tables II and III five typical sets of scores, namely, that of (1) the best school, (2) average of the upper third (9 schools), (3) the ninth school, (4) the average of all the schools (27 schools), (5) the lowest school. These five scores on each test, both for accuracy ("Rights") and speed ("Attempts"), show the range of attainment for representative algebra instruction, the best and the poorest attainment and a tentative standard (for the formal processes) which we would like to set before teachers of algebra.

The average attainment of the upper third of the 27 schools as the tentative standard of attainment in first-year algebra.—There have been many suggestions of late that the average of the entire group be taken as standard practice. We believe that the average of the entire group is too low a standard. It is contributed to by many schools of a very low instructional efficiency. On the other hand, in finding the average of some upper group we set a standard contributed to only by schools that have maintained a high level of instructional efficiency. To take the average of the upper nine schools means to discount the best attainment (of the highest school, by allowing for possible unusual conditions which might lead to the very best score), and yet to set a standard that has actually been exceeded by four, five, or six schools. In case any teacher wishes to compare her scores with those of the ninth school (whose record has been exceeded by eight schools) those scores are included here.

The tables make clear the striking differences in the speed and accuracy with which pupils manipulate different formal operations. The very large percentage of "rights" (of the problems attempted), together with the large number of problems worked per minute in the case of Tests I and VI, lead to the conclusion that present methods of *instruction in the removal of parentheses and in the use of exponents leads to satisfactory automatism of these operations.* Pupils have reduced the manipulation of the paren-

TABLE II

EFFICIENCY IN FIRST-YEAR ALGEBRA (RIGHTS)

Norms of attainment in eleven formal operations as determined by: (1) attainment of the most efficient school, (2) average attainment of the upper third of 27 schools, (3) attainment of the ninth school (i.e., the lowest school of the upper third), (4) average attainment of 27 schools, (5) attainment of the lowest of 27 schools.

	NUMBER OF PROBLEMS WORKED CORRECTLY PER MINUTE					
	Test I	Test II	Test III	Test IV	Test V	Test VI
Score of most efficient school................	13.15	5.39	3.24	4.05	1.42	11.84
Average of scores of upper third of 27 schools......	10.95	4.79	2.73	3.50	0.84	10.66
Score of ninth school.....	9.97	4.40 (18 schools were below this attainment)	2.44	2.94	0.53	9.85
Average of 27 schools was	9.65	3.81	2.20	2.85	0.54	9.19
Score of poorest school was	6.12	2.15	1.23	1.71	0.17	4.28

	Test VII	Test VIII	Test IX	Test X	Test XI
Score of most efficient school................	2.75	1.97*	5.28	2.76	0.86
Average of scores of upper third of 27 schools	2.02	4.16*	3.91	1.60	0.73
Score of ninth school......	1.73	5.92* (18 schools were below this attainment)	3.35	1.19	0.62
Average of 27 schools.....	1.37	7.91*	2.76	1.09	0.63
Score of poorest school....	0.51	1.97*	0.12	0.10	0.25

* Minutes required to solve one problem.

thesis to a high state of efficiency. Even in the case of exponents, however, 15 per cent of errors are made on the average.

On the other hand, the score of the poorest school was but slightly more than half that of the average of the upper nine. In fact, in all remaining tests, the score of the poorest school is *less than half* that of the average of the upper third. In some it is only one-fifth to one-sixth of the standard score. Such low levels, combined as they generally are with low efficiency in "translation"

TABLE III

SPEED IN WORKING FORMAL PROBLEMS (AS SHOWN BY NUMBER OF PROBLEMS ATTEMPTED)

Norms of attainment in eleven formal operations as determined by: (1) attainment of the most efficient school, (2) average attainment of the upper third of 27 schools, (3) attainment of the ninth school, i.e., the lowest school of the upper third, (4) average attainment of 27 schools, (5) attainment of the lowest of 27 schools.

	NUMBER OF PROBLEMS WORKED CORRECTLY PER MINUTE										
	Test I	Test II	Test III	Test IV	Test V	Test VI	Test VII	Test VIII	Test IX	Test X	Test XI
Score of most efficient school	14.54	6.26	4.23	4.57	3.00	13.36	3.47	1.44*	7.27	4.19	1.08
Average of scores of upper third of 27 schools	11.59	5.80	3.50	4.14	1.51	12.46	2.88	2.50*	5.80	3.23	0.95
Score of ninth school	10.68	5.51 (18 schools were below this attainment)	3.24	3.87	1.16	11.51	2.63	3.87* (18 schools were below this attainment)	5.23	2.82	0.83
Average of 27 schools	10.36	4.94	3.02	3.60	1.14	11.16	2.35	3.05*	4.85	2.71	0.81
Score of poorest school	6.50	2.82	2.20	2.40	0.49	7.91	1.40	1.44*	2.88	1.48	0.47

* Minutes required to do one problem.

processes (see Table III), certainly *call for closer supervision of instruction.* As pointed out in last year's report, *they indicate, in some cases at least, a lack of teaching emphasis on certain operations with a corresponding greater emphasis on others.* This question of teaching emphasis in the case of very good and very poor schools is further illuminated by tracing the record of individual schools in each of the different operations. If this is done, it will be found that the three poorest schools fall in the lowest third in all but 2 tests. The two best schools fall in the upper third in all but 2 of the 11 tests.

The scores for all remaining tests (except Tests I, II, and VI) point out definite lack in making automatic a skill necessary to success in using algebraic methods in original solution. Note, for example, the original scores for Test V, fractions; Test VII, quadratics; Test VIII, graphs; Test IX, radicals; Test X, practical formulae; Test XI, simultaneous equations. The upper group in each case shows what may be regarded as a fair degree of speed, but almost universally shows a woeful lack of accuracy. We believe that *instruction in these operations should lead to automatic efficiency. Under such a condition recurring mistakes in a class would be impossible, accidental mistakes would occur, but be very rare* (e.g., mistakes in reading, writing, and rare mistakes in arithmetic). The test at least reveals the results of instruction in these processes.

2. THE RELATION BETWEEN EFFICIENCY IN THE AUTOMATIC PROCESSES AND EFFICIENCY IN REASONING PROCESSES

Teachers and administrators who have been loath to adopt "measuring methods" in their school practice and who have hesitated to introduce practice exercises do so on the ground that such a procedure will "mechanize" teaching. This same skepticism has been commonly expressed as a result of the 1915 report on algebra—the point of view of the writer and the implications of the report certainly having been misunderstood. The 1916 testing was, therefore, planned to take account of this question. Seventeen school systems took the eleven formal tests and also the verbal or reasoning tests. Table IV gives the comparative ranks of each of the 17 schools in both types of test. To permit a composite comparison of ability of the two types, the rank of each of the 17 schools has been determined for accuracy in each of the eleven tests. A tabulation was then made to give the aggregate ranks of each school. The list was then reranked, calling the school that had the lowest number of aggregate ranks 1, the next lowest 2, etc., throughout the list. The final comparison of ranks is shown in Table IV.

TABLE IV

COMPARISON OF THE RANK OF 17 SCHOOLS IN EFFICIENCY IN MANIPULATION OF THE FORMAL OPERATIONS WITH RANK OF SAME SCHOOLS IN EFFICIENCY IN REASONING TESTS

School System	Rank in Aggregate Scores Made in:		School System	Rank in Aggregate Scores Made in:	
	Reasoning Tests	All Formal Tests		Reasoning Tests	All Formal Tests
A..........	1	1	π..........	3	6
C..........	9	9	M..........	8	11
F..........	11	5	N..........	4	14
Y..........	16	16	P..........	13	2
G..........	14	10	Q..........	6	4
I..........	10	13	R..........	12	7
J..........	2	3	V..........	15	17
Z..........	7	8	W..........	17	15
K..........	5	12			

The first, ninth, sixteenth, tenth, second, seventh, third, eighth, sixth, fifteenth, and seventeenth schools in the "reasoning tests" occupy ranks in the "formal tests" equal to these or not displaced by more than 3 places in 17. That is, 11 of the 17 schools stand almost in the same relative position in reasoning efficiency as in formal efficiency. Of the whole 17 schools only 2 show a distinct reversal of position in the two types, schools N and P. In view of the preponderance of evidence showing high positive correlation between these two types of processes we are unable to account for the status of the data on these two schools. Of the remaining 4 schools, G is displaced four places, R, 5 places, and K, 7 places. For those interested in correlations we may say that by Spearman's footrule method (with 17 cases any coefficient has little validity, hence we may as well use a rough approximate method as the more accurate product-moment method) $r = 0.59$, which may be regarded as a high correlation.

Careful study of our complete data leads us to the conviction that those teachers who have been most successful in developing skill in manipulating the formal processes have also been most successful in developing ability to "translate" verbally stated problems into algebraic symbolism. We regard this as pertinent to the question of "mechanization" of the teaching process. . . . . We believe that it emphasizes the main point we are trying to make: namely, that with carefully planned practice periods, using scientifically built-up practice material, teachers will be still more effective in developing "original ability."

Certainly, without the aid of scientifically planned practice devices, teachers who have been most successful in doing the one, have been, very generally, most successful in doing the other. We have no reason to believe, from our present detailed data, that teachers who drill their pupils to a relatively high state of "formal efficiency" do so by neglecting the higher thought processes in algebraic solution. The teachers, furthermore, have done this without the aid of carefully differentiated drill exercises. They have followed, in the main, the traditionally planned textbook organization of theoretical presentation and "problem exercises."

We take the position that much greater correlation might come about, then, through the aid of carefully designed practice exercises which put the chief emphasis in practice upon the difficult processes, saving time for more detailed training in those types of solution that call for analysis discrimination and other higher thought processes. Testing pupils' ability to manipulate algebraic operations does not lead to a general blind "mechanization" of the teaching process. It leads to an intelligent understanding of *what* operations ought to get the most drill and relatively how much they ought to be drilled—in a word, *differentiated drill takes the place of blind wholesale drill such as will ensue if one follows the textbook organization as the sole guide.*

3. THE RELATION BETWEEN SPEED AND ACCURACY IN "FORMAL EFFICIENCY"

Table V gives the rank of each of the schools in number of problems attempted and number right. The high degree of relationship between speed (represented by "number of problems

TABLE V

Comparison of Speed and Accuracy, as Shown by Rank of 27 Schools in Each of 11 Formal Tests in Number of Problems Attempted and Number of Problems Right

School System	Test I		Test II		Test III		Test IV		Test V	
	Rank in No. Problems									
	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.
A	14	12	15	13	2	1	17	8	5	5
B	4	3	3	3	20.5	15	2	1	3	3
C	15	14	16	22	18.5	13	22	21	22.5	14
D	2	4	13	11	6	4	4	5	10	9.5
E	17	13	27	27	22.5	20	16	19	21	21
F	12	9.5	7.5	7	9	5	15	17	15.5	16
G	20	18	6	2	11	9	26	23	11	8
H	5	2	2	1	4	3	18	11	27	26
I	27	27	9	10	3	2	8	13	7	7
J	88	8	7.5	5	12	12	7	6	1	1
K	22	22	25.5	23	18.5	23	21	18	22.5	23
L	7	4.5	11	8	5	6	1	2	18	21
M	23	24	12	17	25	26	14	15.5	12	19
N	10	19	22	25	22.5	25	12	15.5	17	15
O	26	25	25.5	24	27	24	20	20	24	21
P	19	16	20	14	15	11	5	4	9	4
Q	1	1	1	4	7	18.5	6	10	4	6
R	18	17	4	15	16	18.5	11	14	2	12
S	6	6	5	6	14	7	3	3	15	11
T	25	23	24	20	1	14	27	27	14	18
U	16	20	21	21	8	8	10	9	8	9.5
V	24	26	17	19	26	27	25	26	26	25
W	13	15	19	18	17	22	23	25	25	27
X	9	11	14	9	10	10	24	24	13	24
Y	21	21	23	26	20.5	21	19	22	19	17
Z	3	5	10	12	11	17	9	12	20	13
π	11	7	18	16	24	16	13	7	6	2

	Test VI		Test VII		Test VIII		Test IX		Test X		Test XI	
	Rank in No. Problems											
	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.
A	18	6	11	2	19	11	5	2	23	6	10	9.5
B	2	27	1	1	2	6	1	1	1	1	2	1
C	9	12	23	23.5	16	16	16	12	2	7.5	25	23
D	4	4	18	14	24	26	12	7	18	14	12	12
E	24	22	25	27	22	2	24	26	12.5	15	23	19.5
F	15	13	10	7	12	13	17	16	9	9	1	4
G	25	23.5	27	25.5	26	22	23	14	21	18	20.5	11
H	26	25	9	4	10	7	3	3	12.5	16.5	14	7.5
I	23	16	24	20	9	9	6	8	25	11	20.5	18
J	6	8	7	13	5	5	11	10	6	16.5	5	7.5
K	7	7	4	8	1	1	25	24	26	26	9	13
L	8	9	8	15	18	4	2	9	10	24	8	21

TABLE V—*Continued*

School System	Test VI		Test VII		Test VIII		Test IX		Test X		Test XI	
	Rank in No. Problems											
	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.	Att.	Rt.
M......	14	10	5	17	4	12	18	19	5	10	7	15
N......	10	11	20	10	20	15	26	25	24	23	16.5	16.5
O......	22	20	19	18	25	23	10	6	19	5	27	27
P......	13	5	16	5	6	10	13	4	8	4	11	6
Q......	12	23.5	3	6	15	19	9	5	16	19	4	2
R......	17	14	15	12	8	6	14	18	7	12.5	6	3
S.......	1	3	2	3	13	18	22	20	3	2	20.5	14
T......	21	21	13	9	243	8	15	13	20	12.5	16.5	19.5
U......	20	15	17	16	14	24	20	22	15	7.5	24	25
V......	27	26	26	21	....	....	19	23	22	21	26	24
W......	16	19	21.5	22	17	25	8	17	17	25	14	22
X......	11	18	6	11	7	20	7	11	4	22	3	5
Y......	19	17	21.5	23.5	11	14	....	....	......	......	20.5	16.5
Z.......	3	2	13.5	19	21	21	4	15	14	20	18	22
π.......	5	1	13.5	25.5	23	17	21	21	11	3	14	9.5

attempted") and accuracy (represented by "number of problems right") will be evident to the reader. The schools ranking high in speed are very generally the schools ranking high in accuracy. In Test V, which we will take at random, the footrule correlation gives $R=0.70$ or $r=0.89$ by Spearman's transmutation formula. By supplying the detailed data in Table V we do not need to discuss the material in detail here. The conclusion is clear that speed and accuracy in these formal operations go hand in hand.

We feel that this point is of great importance in the methodology of teaching. From individual experimentation in the classroom, the writers are convinced that *speeding up class work conduces to greater accuracy in the manipulation of formal operations.* The results of testing pupils in 27 schools confirm our classroom experience.

Thus the teaching program that we would lay before teachers is contributed to by the findings of the last sections: First, *drill pupils daily in the formal operations by stressing the most difficult and the most important processes.* (As a result of our investigation we know the most difficult processes, and they will be taken up in the next section. We do not know the *most important processes* however. The Illinois Committee on Standards has set as one of its tasks for the next year of research the determination of the answer to this question.) Secondly, drill pupils under "time conditions." *It should be made "second nature" to manipulate the different formal operations under the pressure of "time." Life situations continually demand it—the school should constantly prepare for it.* Again classroom experimentation of blackboard and seat work done by the writers leads to this conclusion.

At this point let us call to mind the steps of our general program. We have stated the aim and outcome of teaching algebra; classified the content carefully into fundamental operations and types of problems; designed tests for each fundamental operation and for verbal problems, establishing definitely our principles of design and procedure; given the tests to large groups of pupils; set up tentative standards of speed and accuracy for each operation and made it possible to compare the practice and efficiency of a teacher or a school with the efficiency of the "best schools"; discussed the "yardstick" function of tests, the relation between formal efficiency and reasoning ability, and the *relation* between speed and accuracy.

4. THE CHIEF FUNCTION OF STANDARDIZED TESTS IN FIRST-YEAR ALGEBRA: DIAGNOSIS

We said that tests have two functions: (1) measurement of abilities in order to establish norms and compare attainments; (2) diagnosis of difficulties of individual pupils. The greater of these two functions is the latter.

We are primarily interested in ways and means of improving instruction in first-year algebra—only incidentally interested in establishing standards or norms of attainment. The norm of attainment is of value only as it spurs the teacher to a high standard of class-teaching; the test is of value only as it points out principal typical difficulties that pupils face in learning algebra. If it is properly designed, that is, if it measures *specific processes*, it can be made to do this.

Pupils' errors in first-year algebra.—Pupils make two principal kinds of errors: (1) accidental errors (i.e., errors of reading, writing, following directions, arithmetic, etc.), and (2) recurring errors.

Recurring errors supply a means of determining exactly which types of problems, operations, or processes pupils have not mastered provided the tests are rigidly designed on the "cycle" or "rotation" principle. Thus, in diagnosing difficulties in "learning," *it is the recurring errors that are significant.* They reveal which types of operation have not been mastered. The *Standardized Tests* submitted herewith, designed as they are on the principle of recurring types of operation *reveal exactly which are the typical difficulties pupils face.*

Table VI gives a complete list of recurring errors made by 100 pupils on each of the 11 formal operations, together with the frequency of occurrence and the percentage that the recurring errors of each particular type are of all recurring errors made.

This list is complete for the pupils whose work was studied. It has been compiled by the writers of this report, who scored and judged carefully every problem on each of the 11 tests for each pupil. In many cases the combined judgment of the two writers was made on particular errors. The list can be used by teachers as a definite guide to help eliminate these particular errors on the part of their pupils. Knowledge as to which are the most difficult operations will lead to wise methods of eliminating the errors in such operations.

Let us summarize *the outstanding facts concerning the recurring errors made on formal operations.*

Test I. *Removal of parentheses.*—A fairly high degree of efficiency was maintained. Of the mistakes that were made, one-half were in the use of signs. Of these, twice as many were made when the $-$ sign was used outside the parentheses as when the $-$ sign precedes the first letter inside (with $+$ sign outside). (The writers' drill exercises take account of this.) One-sixth of all pupils make an arithmetic mistake on this problem.

Test II. *Special products.*—One-fourth of all recurring mistakes are on cross-products. This is also confirmed by predominance of error No. 14 in Test IV. A very large group of pupils fail to square the literal factor when the exponent is greater than 1, and another considerable group use the product of two numbers instead of twice the product.

TABLE VI

Complete List of "Recurring Errors" for Each Test, Together with Frequency and Percentage of Occurrence

Each error in this list has occurred more than once upon the paper of some pupil. The numbers in the first column opposite each error, indicate the *number of pupils* [out of a "random selection" of 100] who made that particular error *one or more* times in the test. The numbers in the second column indicate the percentage of all pupils who make "recurring errors," who made that particular error.

No. of Error	Tests	I No. Pupils Making Error One or More Times	II Percentage of All "Recurring Errors"
	TEST I		
1	Arithmetic errors: $24\cdot 8=182$; $18\cdot 0=18$	15	35.7
2	Omission of symbols, signs, or terms: $6(3x+8)=18+48$	6	14.3
3	Errors of reading or writing: $(2x-3)^2=4y^2-12x+9$	....	....
4	Errors in use of signs in multiplication		
	a) Minus sign preceding ( ): $-4(3x-4)=-12x-16$	14	33.3
	b) Minus sign preceding first term within ( ): $9(-7x-1)=63x-9$	7	16.7
	TEST II		
1	Arithmetic errors	5	6.6
2	Omission of symbols, signs, or terms	1	1.3
4	Errors in use of signs in multiplication	10	13.4
5	Errors in use of signs in addition: $(x+3)(x-4)=x^2+x-12$	2	2.6
6	Errors in cross-products: $(a-4)(a+5)=a^2-20$	20	26.7
	(NOTE.—It may be due to carrying over rule for product of sum and difference; made by pupils who got product of sum and difference correctly.)		
7	Use of product of two numbers instead of twice the product: $(2x-3)^2=4x^2-6x+9$	9	12.0
8	Inability in specific operation tested	15	20.0
9	Failure to square literal factor with exponent greater than 1: $(a^2+3)(a^2-3)=a^2-9$	13	17.4
	TEST III		
1	Arithmetic errors	33	47.8
8	Inability to grasp principle of substitution	3	4.4
16	Squaring product of literal factor instead of the one designated: $ab^2=(3\cdot 2)^2=36$ if $a=3$, $b=2$	22	31.8
11	Using exponent as coefficient and vice versa: $x^2=10$ if $x=5$	2	2.9
12	Adding factors instead of multiplying them: $3xy=3+5+2$ if $x=5$, $y=2$	9	13.1

TABLE VI—*Continued*

No. of Error	Tests	I No. Pupils Making Error One or More Times	II Percentage of All "Recurring Errors"
	TEST IV		
2	Omission of factors	4	4.3
3	Errors in writing	7	7.4
4	Errors in use of signs in multiplication	2	2.1
8	Positive inability to factor	37	39.4
13	Incorrect division by monomial factor: $5x^2+15x^3=5x^2(3x)$	1	1.1
14	Factors such that the sum of the cross-products are incorrect: $5x^2+16x+3=(5x+3)(x+1)$	20	21.3
15	Factors give correct first and second term, but incorrect third: $a^2+10a+24=(a+8)(a+2)$	4	4.3
16	Failure to recognize monomial factor: $6a^2+9a^3=(2a+3)(3a+3a^2)$	14	14.8
17	Failure to get *highest* monomial factor: $12y+18y^4=2(y+9y^4)$	5	5.3
	TEST V		
1	Arithmetic errors	28	18.4
1*a*	Arithmetic error of this type: $-n\cdot 0=n$	8	5.2
3	Error in writing	3	1.9
4	Errors in use of signs in multiplication	4	2.6
5	Errors in use of signs in addition	28	18.3
8	Inability in specific operation tested	27	17.7
18	Failed to change signs if numerator preceded by − sign: $\frac{4x-2}{3}-\frac{x-3}{4}=0$; $16x-8-3x-9=0$	27	17.7
19	Multiplied only one side of equation by L.C.M.: $\frac{3x}{4}+\frac{3x}{2}=5$; $3x+6x=5$	16	10.5
20	Error in use of sign in transposition: $3x+2=6$; $3x=6+2$	4	2.6
22	Error in division (result inverted, e.g., $13x=-1, x=\frac{-13}{1}$)	1	.6
23	Multiplied numerator by L.C.D. (e.g., $\frac{2}{2-x}-\frac{5}{2+x}=0$; $4-2x-20+5x^2=0$)	7	4.6
	TEST VI		
1	Arithmetic errors	11	8.0
3	Error in writing	6	4.4
8	Inability in specific operation tested	21	15.3
25	Incorrect use of exponents in multiplication:		
	a) Multiplying exponents instead of adding them: $a^3\cdot a^5=a^{15}$	9	6.5
	b) Subtracting exponents instead of adding them: $x^7\cdot x=x^6$	4	2.9
	c) Failure to add $x^7\cdot x=x^7$	1	0.7

TABLE VI—*Continued*

No. of Error	Tests	I No. Pupils Making Error One or More Times	II Percentage of All "Recurring Errors"
	TEST VI—*Continued*		
26	Incorrect use of exponents in involution		
	a) Adding instead of multiplying: $(x^3)^4=x^7$	45	32.4
	b) Subtracting instead of multiplying: $(x^3)^4=x$	1	0.7
	c) Dividing instead of multiplying: $(x^3)^4=x^{\frac{4}{3}}$	3	2.2
	d) Raising to a power instead of multiplying: $(x^3)^2=x^9$	18	13.1
	e) Failure to raise all possible factors to required power: $(ab^2)^2=(ab^4)$	16	11.6
27	Incorrect use of exponents in division		
	a) Adding instead of subtracting: $\frac{c^3}{c^2}=c^5$	1	0.7
	b) Multiplying instead of subtracting: $\frac{y^5}{y^4}=y^{20}$	2	1.5
	TEST VII		
8	Inability in specific operation tested	32	21.6
28*a*	Find only one root: $x^2-81=0$; $x=9$	28	19.0
b	One root correct; other incorrect: $y^2+y=6$; $y=-3$ or -2	2	1.3
14	Incorrect solution due to: factors such that sum of cross-products is incorrect: $n^2-7n=12$; $n=6$ or 2	71	48.0
29	Incomplete, e.g., $x^2-81=\quad$, $x-9=0$, $x+9=0$	9	6.0
30	Incorrect use of formula method of solution	5	3.4
31	Solution by factoring left member; right member not 0: $n^2-7n=12$, $n(n-7)=12$	1	0.7
	TEST IX		
1	Arithmetic error	14	9.7
8	Inability in specific operation tested	88	60.6
24	Incomplete: $\sqrt{8}=\sqrt{4}\times\sqrt{2}$	2	1.4
32	Left factor of same power as degree of radical: $\sqrt{32}=2\sqrt{8}$; $\sqrt{a^3b^4}=b^2\sqrt{a^3}$	23	15.9
33	Error in factoring; perfect square factor correct, other incorrect: $\sqrt{32}=4\sqrt{4}$; $\sqrt{50}=5\sqrt{5}$	4	2.8
34	Does not multiply numerator by rationalizing factor: $\sqrt{\frac{3}{5}}=\frac{1}{5}\sqrt{3}$	8	5.5
35	Failed to take square root although factoring correctly: $\sqrt{x^5y^8}=x^4y^8\sqrt{x}$	2	1.4
36	Factored correctly, took square root correctly; placed radical sign over wrong factor: $\sqrt{a^5b^4}=\sqrt{a^2b^2}\cdot a$	4	2.8

TABLE VI—*Continued*

No. of Error	Tests	I No. Pupils Making Error One or More Times	II Percentage of All "Recurring Errors"
	TEST X		
3	Error in writing	4	2.5
8	Inability in specific operation tested	12	7.5
20	Errors in use of signs in transposition	4	2.5
22	Error in division, result inverted: e.g., $V=lwh$, $w=\frac{hl}{V}$	44	27.3
24	Incomplete	1	0.6
37	Transposing factors instead of terms: $V=Lwh$; $w=V-Lh$	12	7.5
38	Interchanged factors of both sides of equation: $E=\frac{PL}{K}$; $P=\frac{EL}{K}$	30	18.6
39	Error in selecting coefficient of the unknown in $L=\frac{Mt-g}{t}$, $M=\frac{Lt}{t-g}$ (considers $(t-g)$ as coefficient of M)	51	31.6
40	Divides a term of numerator by the denominator: e.g., $L=\frac{Mt-g}{t}=M-g$	3	1.9
	TEST XI		
1	Arithmetic errors: $3\cdot 21=73$	25	50
1*a*	Arithmetic errors of the type $n\cdot 0=0$	1	2
3	Errors in writing	4	8
8	Inability in specific operation tested	3	6
19	Multiply only one side of equation by L.C.M.	2	4
20	Error in use of signs in transposition: $6x-4=10$; $6x=6$	3	6
23	Multiply numerator by denominator in clearing fraction	1	2
24	Incomplete (e.g., $-y=-3$)	2	4
41	Errors in use of signs in subtraction: $\begin{cases}10x-15y=65\\10x+6y=78\end{cases}$; $-9y=-63$	4	8
42	Errors in use of signs in division: $-12x=24$; $x=2$	1	2
43	Adds left member, subtracts right member or vice versa: $\begin{cases}4x+2y=20\\3x-2y=1\end{cases}$; $7x=19$	3	6
44	Adds corresponding members when he should subtract: $\begin{cases}2p+10q=54\\2p-\ 3q=2\end{cases}$; $7q=52$	1	2

We may dispose once for all, of the error marked No. 8 in each test, "positive inability in the specific operation tested."

We give below the percentage of all errors that were made, which show inability to use the operation in question in each test.

Tests	Percentage	Tests	Percentage
2	20	7	21.6
3	4.4	9	60.6
4	39.4	10	7.5
5	17.7	11	6
6	15.3		

This table reveals that there is a very large number of our pupils that are absolutely unable to handle certain combinations of the following operations: special products, factoring (complete quadratic), fractions, exponents, quadratics, and radicals. This percentage, it should be noted, may be deceptive, for in some cases it includes only a small proportion of all errors made. It is striking at least as showing the cases in which positive inability to do the problems stood out.

Test III. *Substitution.*—Relatively few errors. Of those that were made, nearly one-third are due to "squaring product of literal factors instead of one designated." Practically half of all errors are arithmetic.

Test IV. *Factoring.*—Thirty-nine per cent of all recurring errors show positive inability to factor particular types of factoring problem (mostly complete quadratics). The outstanding errors are found to be error numbers 14, "factors such that sum of cross-products is incorrect," and 16, "failure to recognize monomial factor."

Test V. Errors in the use of signs again predominate, e.g., in addition and in multiplication failure to change signs if numerator is preceded by − sign. Also need for emphasis on multiplying both sides of equation by L.C.M.

Test VI. There are three outstanding errors in the use of exponents—incorrect use in involution (adding instead of multiplying, raising to power instead of multiplying, and failure to raise to required power). There are other scattering errors which recur rather frequently.

Test VII. Pupils are prone to find only one root. Their most outstanding mistake again is in cross-products—getting factors such that sum of cross-products is incorrect.

Test IX. Sixty per cent of all recurring errors show positive inability to handle particular kinds of radicals, e.g., of the type $\sqrt{\frac{2}{3}}$ especially. Of the other recurring errors, pupils commonly leave factor of same power as degree of radical. They also do not multiply numerator by rationalizing factor.

Test X. Three typical recurring mistakes: (1) errors in division, results inverted, e.g., in $v=lwh$, $w=\frac{hl}{v}$; (2) interchanging both sides of the equation; (3) errors in selecting coefficient of the unknown, e.g., in $L=\frac{Mt-g}{t}$, $M=\frac{Lt}{t-g}$, pupil considers $(t-g)$ as coefficient of M.

Test XI. Aside from purely arithmetic errors, the solution of simultaneous equations of the type represented here has been rather well automatized. Errors in use of signs stand out again. We would like to call particular attention to the frequency of recurrence of errors in the use of signs in all tests where they are possible.

Methods of "differentiated" drill will improve instruction in the formal processes.—The giving of standard tests is only a first step in the program of improving instruction. The tests, however, if corrected by the teacher will lead to definite knowledge as to the types of problems upon which she needs to drill her pupils. Great insight can be obtained by the teacher into the difficulties of her pupils if she will take the trouble to examine the tests carefully and classify the mistakes made. Typical recurring ones will stand out clearly and these she can drill upon *in a differentiated way.*

The working out and giving of the Standardized Tests have offered the writers an opportunity to extend their service to teachers of algebra with a thoroughgoing and complete classification of the recurring errors made by pupils. The tabulation and discovery of errors are being followed up by the careful design of drill exercises to aid in the elimination of the weaknesses in learning. This in turn is being supplemented by thorough experimentation to determine the frequency with which particular operations should be drilled, the distribution of the different types of arrangement in the different exercises, best ways of presenting material, etc. Thus the fifth step in our program has just been taken.

We are now organizing a thoroughgoing experiment in the Parker High School, Chicago, and in a group of co-operating high schools in and about Chicago, in the attempt to do the things outlined above. Practice exercises have been organized in a tentative way, separate exercises for each operation.[1] Those exercises have emphasized the most frequently recurring errors roughly in proportion to their difficulty.

From this point on real progress in improving algebra instruction can be made only by teachers of the subject working in their classrooms. We urge, therefore, the undertaking of a very thorough co-operative experiment by teachers of algebra along the lines marked out by this report. We wish to enlist the co-operation of all teachers interested in the attempt to improve the teaching of algebra along the lines suggested herewith. A definitely worked out experimental plan of trying out various ways of teaching algebra will be mailed to anyone interested enough to take part in a co-operative way. Complete records of the conduct of the experimental work will be kept and reported in the pages of this journal next year.

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[1] Sample sets of practice exercises with prices may be secured from H. O. Rugg, School of Education, University of Chicago. They will be sold at the cost of publication and handling.